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UNIFORM CONVEXITY IN LORENTZ SEQUENCE SPACES

BY

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ABSTRACT

Necessary and sufficient conditions for Lorentz sequence spaces $d(a,p)$ $(1 < p < \infty)$, to be uniformly convexifiable are given. In case $p \geqq 2$ the modulus of convexity is calculated.

Let $1 \leqq p < \infty$, for any $a = \{a_1, a_2, \cdots\} \in c_0 \setminus l_1$ $1 = a_1 \geqq a_2 \geqq \cdots \geqq 0$, let

$$d(a,p) = \{x = \{\alpha_i\} \in c_0; \quad \|x\| = \left(\sup_{\sigma \in \pi} \sum_{i=1}^{\infty} |\alpha_{\sigma(i)}|^p a_i\right)^{1/p} < \infty$$

where π is the set of all permutations of the natural numbers. The space $d(a,p)$ is a Banach space called Lorentz sequence space. For basic properties of Lorentz sequence spaces we refer the reader to [1,2].

We recall that a Banach space X is called uniformly convex if for every $\varepsilon > 0$ there exists $\delta_X(\varepsilon) > 0$ such that $\delta_X(\varepsilon) = \inf(1 - \|(x+y)/2\|)$, where the infimum is taken over all $x, y \in X$ satisfying $\|x\|, \|y\| \leqq 1$ and $\|x - y\| \geqq \varepsilon$. The function $\delta_X(\varepsilon)$ is called the modulus of convexity of X. A Banach space $(X, \|\ \|)$ is called uniformly convexifiable if there exists an equivalent norm $\|\ \|_1$ such that $(X, \|\ \|_1)$ is uniformly convex.

A necessary and sufficient condition for uniform convexity of Lorentz function spaces was already given by Halperin [4]. We begin by reproducing here the argument of Halperin in the special case of Lorentz sequence spaces. Our first result, Theorem 1, is to a large extent (mainly the equivalence I $\Leftrightarrow$ III) already contained in [4].

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Our main result here is Theorem 2. In it we evaluate (up to a bounded factor) the value of $\delta_X(\varepsilon)$ in terms of $S(n)=\Sigma_{i=1}^{n} a_i$ in the case $p \geqq 2$. Theorem 3 shows (again in the case $p \geqq 2$) that the asymptotic formula we get for the modulus of convexity in Theorem 2 cannot be improved by passing to an equivalent norm. That is, if Y is a space isomorphic to X then $\delta_Y(\varepsilon) \leqq A\delta_X(\varepsilon)$ for some constant A independent of ε. Our final result, Theorem 4, characterizes all the functions $\delta(\varepsilon)$ which are equivalent to the modulus of convexity of some Lorentz sequence space $d(a,p)$ with $p \geqq 2$.

We say that the function $u(x)$ satisfies the Δ_2 condition for large values of x, if there exists an $x_0 > 0$ and a constant $C > 0$ such that $u(2x) \leqq Cu(x)$ for all $x \geqq x_0$. Two functions $f(x)$ and $g(x)$ defined on some set K of reals are called equivalent, (denoted by $f \sim g$), if there exist constants $A, B > 0$ such that $Bg(x) \leqq f(x) \leqq Ag(x)$ for all $x \in K$.

In the sequel we denote by $\{e_n\}_{n=1}^{\infty}$ the natural unit vector basis of the Lorentz sequence space $d(a,p)$. For notions in general Banach space theory we follow the terminology of [6].

THEOREM 1. *Let $d(a,p)$ $(1 < p < \infty)$ be a Lorentz sequence space. The following conditions are equivalent*:

I) *$d(a,p)$ is uniformly convex.*

II) *$d(a,p)$ is uniformly convexifiable.*

III) $\inf_n S(2n)/S(n) = k > 1$.

IV) $S(n)/n \sim a_n$.

The proof of the equivalence III $\Leftrightarrow$ IV is immediate:

III $\Leftrightarrow$ IV: If $\inf_n S(2n)/S(n) = k > 1$, then

$$k - 1 \leqq (S(2n) - S(n))/S(n) \leqq na_n/S(n), \text{ and}$$

hence:

$$a_n \leqq S(n)/n \leqq (k-1)^{-1} a_n.$$

Conversely, if $S(n)/n \sim a_n$, then

$$1 \leqq S(2n)/S(n) \leqq C \cdot 2na_{2n}/na_n,$$

which implies that $a_{2n}/a_n \geqq (2C)^{-1}$, It follows that:

$$(S(2n) - S(n))/S(n) \geqq na_{2n}/Cna_n \geqq (2C^2)^{-1},$$

and thus $S(2n)/S(n) \geqq 1 + (2C^2)^{-1}$.

To prove the equivalences I $\Leftrightarrow$ II $\Leftrightarrow$ III we need some lemmas.

LEMMA 1. *Let $\{x_n\}$ be an unconditional basis of a Banach space X with an unconditional constant 1. Then X is uniformly convex if and only if for every $\theta > 0$ and every $1 > \eta > 0$ there exists a $\delta_1(\eta, \theta) > 0$ such that the following holds: If $x = \Sigma \alpha_i x_i$, $y = \Sigma \beta_i x_i \in X$, $\|x\|, \|y\| \leqq 1$ satisfy $\|\Sigma_{i \in E} \alpha_i x_i\| \geqq \theta$ where $E = \{i; (1-\eta)|\alpha_i| \geqq |\beta_i|\}$ then $\|(x+y)/2\| \leqq 1 - \delta_1(\eta, \theta)$.*

PROOF. Suppose X is uniformly convex and let $\delta_X(\varepsilon)$ be its modulus of convexity. If x and y are as in the statement of the lemma then

$$\|x - y\| \geqq \|\sum_{i \in E} (\alpha_i - \beta_i) x_i\| \geqq \eta \|\sum_{i \in E} \alpha_i x_i\| \geqq \eta\theta$$

and hence $\|(x+y)/2\| \leqq 1 - \delta_X(\eta\theta)$. To prove the converse, notice that it is enough to show that for every $\varepsilon > 0$ $\inf(1 - \|(x+y)/2\|) > 0$ where the inf is taken over all $x = \Sigma \alpha_i x_i$, $y = \Sigma \beta_i x_i$ satisfying $\|x\| \leqq 1, \|y\| \leqq 1, \|x - y\| \geqq \varepsilon$ and α_i, $\beta_i \geqq 0$ for all $i = 1, 2, \cdots$. To see this, let

$$A = \{i; \alpha_i \beta_i < 0\} \quad B = \{i; i \in A \quad |\alpha_i| \geqq |\beta_i|\} \text{ and}$$

$$C = \{i; i \in A, \ |\alpha_i| < |\beta_i|\}.$$

Define

$$\alpha'_i = \begin{cases} 0 & i \in C \\ |\alpha_i| & \text{otherwise} \end{cases} \qquad \beta'_i = \begin{cases} 0 & i \in B \\ |\beta_i| & \text{otherwise} \end{cases}$$

and $x' = \Sigma \alpha'_i x_i, y' = \Sigma \beta'_i x_i$. It is easily checked that $\|x'\|, \|y'\| \leqq 1, \|x + y\| \leqq \|x' + y'\|$ and $\|x' - y'\| \geqq \|x - y\|/2$.

Assume now that a suitable $\delta_1(\eta, \theta)$ exists and that x and y are vectors with norm 1 satisfying $\|x - y\| > \varepsilon$ and having non-negative coefficients α_i, β_i. Put $G = \{i; \alpha_i \geqq \beta_i\}$ and $F = \{i; \alpha_i \leqq \beta_i\}$. Since $\|x - y\| \geqq \varepsilon$ either $\|\Sigma_{i \in G} (\alpha_i - \beta_i) x_i\| \geqq \varepsilon/2$ or $\|\Sigma_{i \in F} (\alpha_i - \beta_i) x_i\| > \varepsilon/2$, and we may clearly assume that the first case holds. Let $H = \{i, (1 - \varepsilon/4)\alpha_i \geqq \beta_i\}$. Then for all $i \in G \setminus H$, $\alpha_i - \beta_i \leqq \varepsilon \alpha_i/4$ and hence $\|\Sigma_{i \in G \setminus H} (\alpha_i - \beta_i) x_i\| \leqq \|\varepsilon/4 \, \Sigma_{i \in G \setminus H} \alpha_i x_i\| \leqq \varepsilon/4$. Consequently $\|\Sigma_{i \in H} \alpha_i x_i\| \geqq \|\Sigma_{i \in H} (\alpha_i - \beta_i) x_i\| \geqq \varepsilon/4$.

From the definition of $\delta_1(\eta, \theta)$, it follows that $\|(x+y)/2\| \leqq 1 - \delta_1(\varepsilon/4, \varepsilon/4)$ and hence X is uniformly convex.

LEMMA 2. *Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive numbers. Assume that* $\inf_n S(2n)/S(n) = k > 1$ *where* $S(n) = \Sigma_{i=1}^{n} a_i$. *Then*

(1) $S(2^m n)/S(n) \geqq k^m$ $\qquad n, m = 1, 2, \cdots$

(2) $(S(2^m n) - S((2^m - 1)n))/S(n) \geqq (k/2)^m (k-1)$ $\qquad n, m = 1, 2, \cdots.$

PROOF. (1) Obvious, by induction on m.

(2) $(S(2^m n) - S((2^m - 1)n))/S(n)$

$= (S(2^m n)/S(n))\,(S(2^m n) - S((2^m - 1)n))/S(2^m n)$

$\geqq k^m \cdot 2^{-m} \cdot (S(2^{m+1} n) - S(2^m n))/S(2^m n) \geqq (k/2)^m (k-1).$

LEMMA 3. *Let $d(a,p)$ be a Lorentz sequence space with $\inf_n S(2n)/S(n) = k > 1$ and let $\varepsilon > 0$. Let $\{h_i\}_{i=1}^n$ be a decreasing sequence of non-negative reals satisfying:*

$$\sum_{i=1}^{n} h_i a_i \leqq 1. \tag{3}$$

Assume also that $\{h_{i_j}\}_{j=1}^r$ is a subsequence of $\{h_i\}_{i=1}^n$ for which

$$\left\|\sum_{j=1}^{r} h_{i_j}^{1/p} e_j\right\|^p = \sum_{j=1}^{r} h_{i_j} a_j \geqq \varepsilon^p. \tag{4}$$

Then

$$\sum_{j=1}^{r} h_{i_j} a_{i_j} \geqq C\varepsilon^{p/\log_2 k} \tag{5}$$

where C is some constant dependent only on k.

PROOF. Let $A = \{j\,; i_j > 2^{[m]} j\}$, $B = \{j\,; i_j \leqq 2^{[m]} j\}$ where $m = \log_2(4\varepsilon^{-p})/\log_2 k$, and put

$$\beta_j = h_{i_j} - h_{i_{j+1}} \qquad j = 1, 2, \cdots r-1 \qquad \beta_r = h_{i_r}.$$

We have

$$1 \geqq \sum_{i=1}^{n} h_i a_i \geqq \sum_{j=1}^{r} \beta_j \left(\sum_{k=1}^{i_j} a_k\right) \geqq \sum_{j \in A} \beta_j S(i_j) \tag{6}$$

$$\geqq k^{[m]} \sum_{j \in A} \beta_j S(j) \geqq (k^m/2) \sum_{j \in A} \beta_j S(j).$$

Hence $\Sigma_{j\in A}\beta_j S(j) \leqq 2k^{-m} = \varepsilon^p/2$. By (4) we get that $\Sigma_{j\in B}\beta_j S(j) \geqq \varepsilon^p/2$. Consequently

$$\sum_{j=1}^{r} h_{i_j} a_{i_j} = \sum_{j=1}^{r} \beta_j \left(\sum_{k=1}^{j} a_{i_k} \right) \geqq \sum_{j\in B} \beta_j (S(2^{[m]}j) - S((2^{[m]}-1)j)) \tag{7}$$

$$\geqq (k/2)^{[m]}(k-1) \sum_{j\in B} \beta_j S(j) \geqq (k/2)^m (k-1)\varepsilon^p/2 \geqq C\varepsilon^{p/\log_2 k}.$$

LEMMA 4. *Let $\{x_n\}$ be a symmetric basis of a Banach space X. Assume that there exist increasing sequences of integers $\{n_i\}_{i=1}^{\infty}, \{k_i\}_{i=1}^{\infty}$ and a constant M such that $\sup_i \lambda(n_i k_i)/\lambda(n_i) \leqq M$ where $\lambda(n) = \|\Sigma_{i=1}^{n} x_i\|$. Then X is not uniformly convexifiable.*

PROOF. By [3] it is enough to construct k_i dimensional subspaces V_i of X which are uniformly isomorphic to $l_\infty^{k_i}$. Fix i and let

$$v_m = \sum_{j=(m-1)n_i+1}^{mn_i} y_j/\lambda(n_i) \qquad m = 1, 2, \cdots, k_i.$$

For any sequence of scalars $\{\alpha_m\}_{m=1}^{k_i}$ we get

$$\sup_{1\leqq m\leqq k_i} |\alpha_m| \leqq \left\| \sum_{m=1}^{k_i} \alpha_m v_m \right\| \leqq \sup_{1\leqq m\leqq k_i} |\alpha_m| \quad \left\| \sum_{m=1}^{k_i} v_m \right\|$$

$$\leqq \sup_{1\leqq m\leqq k_i} |\alpha_m| \cdot \lambda(n_i k_i)/\lambda(n_i) \leqq M \sup_{1\leqq m\leqq k_i} |\alpha_m|.$$

Hence, $d([v_m]_{m=1}^{k_i}, l_\infty^{k_i}) \leqq M$.

LEMMA 5. *Let $\{a_n\} \in c_0 \setminus l_1$, $a_1 \geqq a_2 \geqq \cdots \geqq 0$. If $\inf_n S(2n)/S(n) = 1$, where $S(n) = \Sigma_{i=1}^{n} a_i$, then for every $\varepsilon > 0$ there exist sequences of integers $\{n_i\}_{i=1}^{\infty}, \{k_i\}_{i=1}^{\infty}$ such that $S(n_i k_i)/S(n_i) \leqq 1 + \varepsilon$.*

PROOF. Fix $\varepsilon > 0$. By hypothesis there exists a sequence n_i such that $S(2n_i)/S(n_i) < 1 + 2^{-i}$. Let $k_i = i$. Then

$$S(k_i n_i)/S(n_i) = 1 + \sum_{l=2}^{i} (S(ln_i) - S((l-1)n_i))/S(n_i)$$

$$\leqq 1 + (i-1)(S(2n_i) - S(n_i))/S(n_i) \leqq 1 + (i-1)2^{-i} \leqq 1 + \varepsilon,$$

provided i is large enough.

PROOF OF THEOREM 1. Clearly I $\Rightarrow$II. If $\inf_n S(2n)/S(n) = 1$, then by Lemmas 4 and 5 $d(a,p)$ is not uniformly convexifiable, hence II $\Rightarrow$ III.

III $\Rightarrow$I. Let $\theta > 0$ and $0 < \eta < 1$ be given. By Lemma 1, it is enough to show that there exists a $\delta_1(\eta,\theta) > 0$ such that if $x = \sum_{i=1}^{\infty}\alpha_i e_i$, $y = \sum_{i=1}^{\infty}\beta_i e_i$, $\|x\|$, $\|y\| \leqq 1$, $\|\sum_{i\in E}\alpha_i x_i\| \geqq \theta$ where $E = \{i;\ (1-\eta)|\alpha_i| \geqq |\beta_i|\}$, we have $\|(x+y)/2\|^p \leqq 1 - \delta_1(\eta,\theta)$. Also as noted in the proof of Lemma 1, we may assume without loss of generality that $\alpha_i, \beta_i \geqq 0$ and $x = \sum_{i=1}^{n}\alpha_i e_i$, $y = \sum_{i=1}^{n}\beta_i e_i$ for some $n < \infty$. Notice that for $1 < p < \infty$ and $0 < \eta < 1$, we have $0 < \eta' < 1$ such that for every $a, b > 0$ $((a+b)/2)^p \leqq (1-\eta')(a^p + b^p)/2$, provided $(1-\eta)a \geqq b$.

Define the sequence $\{h_i\}_{i=1}^n$ by

$$h_i = \begin{cases} (1-\eta')(\alpha_i^p + \beta_i^p)/2 & i \in E \\ ((\alpha_i + \beta_i)/2)^p & i \notin E \end{cases} .$$

We may assume, without loss of generality that $\{h_i\}_{i=1}^n$ is a decreasing sequence. Indeed let σ be the permutation such that $\{h_{\sigma(i)}\}_{i=1}^n$ is decreasing and let $x' = \sum_{i=1}^n \alpha_{\sigma(i)} e_i$, $y' = \sum_{i=1}^n \beta_{\sigma(i)} e_i$, then clearly $\|x'\|, \|y'\| \leqq 1$, $\|x' - y'\| = \|x - y\|$ and $\|x' + y'\| = \|x + y\|$. Now

$$\sum_{i=1}^n h_i a_i + \eta'(1-\eta')^{-1}\sum_{i\in E} h_i a_i$$

$$= \sum_{i\notin E} h_i a_i + (1 + \eta'(1-\eta')^{-1})\sum_{i\in E} h_i a_i$$

$$\leqq \sum_{i=1}^n (\alpha_i^p + \beta_i^p) a_i/2 \leqq (\|x\|^p + \|y\|^p)/2 \leqq 1.$$

Since $h_i^{1/p} \geqq (\alpha_i + \beta_i)/2$ for every i, we get that

$$\sum_{i=1}^n h_i a_i = \|\sum_{i=1}^n h_i^{1/p} e_i\|^p \geqq \|\sum_{i=1}^n [(\alpha_i + \beta_i)/2] e_i\|^p = \|(x+y)/2\|^p$$

and hence

(8)
$$\|(x+y)/2\|^p \leqq 1 - \eta'(1-\eta')^{-1}\sum_{i\in E} h_i a_i .$$

Let $E = \{i_j\}_{j=1}^r$. Since for $i \in E$ we have

$$h_i^{1/p} = ((1-\eta')/2)^{1/p}(\alpha_i^p + \beta_i^p)^{1/p} \geqq ((1-\eta')/2)^{1/p}\alpha_i,$$

it follows that

$$\sum_{j=1}^r h_{i_j}a_j = \|\sum_{i\in E} h_i^{1/p}e_i\|^p \geqq C_1\|\sum_{i\in E}\alpha_i e_i\|^p \geqq C_1\theta^p.$$

Consequently, by Lemma 3,

$$\sum_{j=1}^r h_{i_j}a_{i_j} = \sum_{i\in E} h_i a_i \geqq C_2\theta^{p/\log_2 k}$$

and hence by (8) $\|(x+y)/2\|^p \leqq 1 - C_3\eta'\theta^{p/\log_2 k}$. This proves the existence of a suitable $\delta_1(\eta,\theta)$.

REMARK 1. *All Lorentz sequence spaces $d(a,p)$ $(1<p<\infty)$ are strictly convex.*

PROOF. Let $x = \Sigma\alpha_i e_i$, $y = \Sigma\beta_i e_i \in d(a,p)$ $\|y\| = \|x\| = 1$ and $\|(x+y)/2\| = 1$. Assume that the sequence $|\alpha_i + \beta_i|$ is arranged in decreasing order. Then

$$1 = \|(x+y)/2\|^p = \Sigma(|\alpha_i+\beta_i|/2)^p a_i \leqq (\Sigma|\alpha_i|^p a_i + \Sigma|\beta_i|^p a_i)/2$$

$$\leqq (\|x\|^p + \|y\|^p)/2 \leqq 1.$$

For this inequality to become an equality we must have $\alpha_i = \beta_i$ for all i, that is $x = y$.

THEOREM 2. *Let $d(a,p)$ $(2 \leqq p < \infty)$ be a Lorentz sequence space, satisfying $\inf_n S(2n)/S(n) = k > 1$. Then there exist constants A_p, $B_p > 0$ such that for all $0 < \varepsilon < 1$*

$$B_p(M(\varepsilon^{-p}))^{-1} \leqq \delta(\varepsilon) \leqq A_p(M(\varepsilon^{-p}))^{-1}, \tag{9}$$

where $\delta(\varepsilon)$ is the modulus of convexity of $d(a,p)$ and $M(x)$ is the inverse function of the function $g(x)$ defined by

$$(10)\qquad g(x)=\begin{cases}0 & x=0\\ \inf_n S(2^m n)/S(n) & x=2^m \quad m=0,1,2,\cdots\\ \textit{linearly on each interval of the form } (2^m,2^{m+1}).\end{cases}$$

Before proving the theorem we make some simple observations concerning the function g. Since

$$(11)\qquad g(2^m)=\inf_n S(2^m n)/S(n)\leqq \inf_n S(2^{m+1}n)/S(n)=g(2^{m+1}),$$

$g(x)$ is an increasing function which satisfies $g(2^m)\geqq k^m$. Now since

$$(12)\qquad g(2^m)/2^m=\inf_n (S(2^m n)/2^m S(n))\geqq \inf_n (S(2^{m+1}n)/2^{m+1}S(n))$$

$$=g(2^{m+1})/2^{m+1},$$

we get that $g(2^m)/2^m$ is a decreasing function of m, and therefore $g(x)/x$ is a decreasing function of x on $[1,\infty)$.Hence $M(x)/x$ is an increasing function of x, where $M(x)=g^{-1}(x)$. Moreover

$$(13)\qquad g(2^{m+1})/g(2^m)$$

$$=\left(\inf_n S(2^{m+1}n)/S(n)\right)\Big/\left(\inf_n S(2^m n)/S(n)\right)$$

$$\geqq \inf_n\left(S(2^m 2n)/S(2n)\right)\left(\inf_n S(2n)/S(n)\right)\Big/\left(\inf_n S(2^m n)/S(n)\right)\geqq k>1.$$

Hence $g(2x)/g(x)\geqq k>1$ for all $x>0$. In that case $M(kx)/M(x)\leqq 2$ for all $x>0$, which is equivalent to the Δ_2 condition. In the sequel we shall need also the following lemma.

LEMMA 6. *Let $f(x)$ be a function satisfying the Δ_2 condition for $x\geqq 1$ such that $f(x)/x$ is increasing. Then*

I) *There exists a convex function $u(x)$ which is equivalent to $f(x)$ on $[1,\infty)$.*

II) *There exists a convex function* $v(x)$ *which is equivalent to* $w(x) = (f(x^{-1}))^{-1}$ *on* $(0,1]$.

PROOF. I. Define the function $u(x) = \int_0^x \sup_{0<t\leqq v} f'(t)dv$. Clearly $u(x)$ is convex, and $u(x) \geqq f(x)$ for all $x \geqq 1$. Since the Δ_2 condition is equivalent to the condition $xf'(x)/f(x) \leqq A$ for all $x \geqq 1$ and some constant A, we get that:

$$u(x) \leqq A\int_0^x \left(\sup_{0<t\leqq v} f(t)/t\right)dv \leqq A\int_0^x (f(v)/v)dv \leqq Af(x)$$

for all $x \geqq 1$.

Part II is proved by a similar argument. The function v is defined by

$$v(x) = \int_0^x \sup_{0<t\leqq v} w'(t)dv.$$

Since

$$S(2^m n)/S(n) \geqq g(2^m) \quad \text{for all} \quad n, m = 1, 2, \cdots \tag{14}$$

the proof of Lemma 2 shows that

$$(S(2^m n) - S((2^m - 1)n))/S(n) \geqq g(2^m)2^{-m}(k-1) \qquad n, m = 1, 2, \cdots \tag{15}$$

In view of these inequalities we can reformulate Lemma 3 as follows.

LEMMA 7. *Let* $d(a,p)$ *be a Lorentz sequence space with* $\inf_n S(2n)/S(n) = k > 1$, *and let* $\varepsilon > 0$. *Let* $\{h_i\}_{i=1}^n$ *be a decreasing sequence, and* $\{h_{i_j}\}_{j=1}^r$ *a subsequence of* $\{h_i\}_{i=1}^n$ *satisfy* (3) *and* (4), *then*

$$\sum_{j=1}^r h_{i_j} a_{i_j} \geqq C(M(\varepsilon^{-p}))^{-1} \tag{16}$$

where $M(x) = g^{-1}(x)$, $g(x)$ *is defined by* (10), *and* $C > 0$ *is some constant.*

PROOF. Let A and B be as in the proof of Lemma 3, where m is a real number chosen so that

$$g(2^m) = 4\varepsilon^{-p}. \tag{17}$$

By (14) the same computation as that in (6) shows that $\Sigma_{j\in A}\beta_j S(j) \leqq 2(g(2^m))^{-1} = \varepsilon^p/2$. Hence by (15) the same computation as that in (7) gives

$$\sum_{j=1}^{r} h_{i_j} a_{i_j} \geqq (k-1) g(2^m) \cdot 2^{-m} \cdot \varepsilon^p / 2 \geqq (k-1) 2^{-m} \geqq C(M(\varepsilon^{-p}))^{-1}.$$

We denote by $C_i > 0$ $i = 1, 2, \cdots$ the constants which will appear in the sequel.

Proof of Theorem 2. We show first that $\delta(\varepsilon) \geqq B_p(M(\varepsilon^{-p}))^{-1}$. Let $x, y \in d(a,p), \|x\| = \|y\| = 1$ and $\|x - y\| \geqq \varepsilon$, $0 < \varepsilon < 1$. As we have seen in the proof of Lemma 1, we may assume, without loss of generality, that $x = \sum_{i=1}^{t} \alpha_i e_i$, $y = \sum_{i=1}^{t} \beta_i e_i$ for some $t < \infty$, that α_i, $\beta_i \geqq 0$ for all i, and that $\|\sum_{i \in H} (\alpha_i - \beta_i) e_i\| \geqq \varepsilon/4$ where $H = \{i; (1 - \varepsilon/4)\alpha_i \geqq \beta_i\}$. Let $n = [-\log \varepsilon]$ and $\eta_k = \varepsilon^{1-k/n}/4$, $k = 0, 1, \cdots, n$. Notice that if $(1 - \eta_k) a \geqq b$, $1 > \eta_k > 0$, $a, b > 0$ then

$$((a+b)/2)^p \leqq (1 - \eta'_k)(a^p + b^p)/2 \tag{18}$$

where η'_k is given by

$$\eta'_k = 1 - 2^{1-p}(2 - \eta_k)^p (1 + (1 - \eta_k)^p)^{-1} = C_1 \eta_k^2 + O(\eta_k^3). \tag{19}$$

Define H_k, $k = 0, 1, \cdots, n$ by

$$H_k = \{i; \quad (1 - \eta_{k+1})\alpha_i < \beta_i \leqq (1 - \eta_k)\alpha_i\} \quad k = 0, 1, \cdots, n-1. \tag{20}$$

$$H_n = \{i; \qquad 0 < \beta_i \leqq 3/4 \alpha_i\}$$

and put $\varepsilon_k = \|\sum_{i \in H_k} (\alpha_i - \beta_i) e_i\|$, $k = 0, \cdots, n$. Clearly

$$\sum_{k=0}^{n} \varepsilon_k^p \geqq (\varepsilon/4)^p \quad \text{and} \quad \bigcup_{k=0}^{n} H_k = H. \tag{21}$$

Now either

$$\Big\| \sum_{i \in H_n} (\alpha_i - \beta_i) e_i \Big\| \geqq \varepsilon/8 \tag{22}$$

or

$$\Big\| \sum_{i \in \bigcup_{k=0}^{n-1} H_k} (\alpha_i - \beta_i) e_i \Big\| \geqq \varepsilon/8. \tag{23}$$

If (22) holds, define the sequence $\{h_i\}_{i=1}^{t}$ by

$$h_i = \begin{cases} (1-\eta'_n)(\alpha_i^p + \beta_i^p)/2 & i \in H_n \\ ((\alpha_i + \beta_i)/2)^p & i \notin H_n \end{cases}.$$

As we remarked in the proof of Theorem 1 there is no loss of generality to assume that $\{h_i\}_{i=1}^{t}$ is a decreasing sequence. We have

$$(24)^{*} \qquad \|(x+y)/2\|^p \leqq \Big\|\sum_{i=1}^{t} h_i^{1/p} e_i\Big\|^p$$

$$= \sum_{i=1}^{t} h_i a_i \leqq (\|x\|^p + \|y\|^p)/2 - \eta'_n(1-\eta'_n)^{-1} \sum_{i \in H_n} h_i a_i .$$

Since $\eta_n = 1/4$, η'_n is a constant independent of ε, and since $h_i \geqq (1-\eta'_n)\alpha_i^p/2$ we get that

$$(25) \qquad \Big\|\sum_{i \in H_n} h_i^{1/p} e_i\Big\|^p \geqq C_2 \Big\|\sum_{i \in H_n} \alpha_i e_i\Big\|^p \geqq C_3 \varepsilon^p .$$

Using Lemma 7 with $\{h_i\}_{i \in H_n}$ as $\{h_{i_j}\}_{j=1}^{r}$, we deduce that $\Sigma_{i \in H_n} h_i a_i \geqq C_4(M(\varepsilon^{-p}))^{-1}$, and by (24) $\|(x+y)/2\| \leqq 1 - C_5(M(\varepsilon^{-p}))^{-1}$. If (23) holds, define the sequence $\{h_i\}_{i=1}^{t}$ by

$$h_i = \begin{cases} (1-\eta'_k)(\alpha_i^p + \beta_i^p)/2 & i \in H_k \quad k = 0,1,\cdots,n-1. \\ ((\alpha_i + \beta_i)/2)^p & \text{otherwise} \end{cases}$$

and assume that $\{h_i\}_{i=1}^{t}$ is a decreasing sequence. Clearly

$$(26) \qquad \|(x+y)/2\|^p \leqq \Big\|\sum_{i=1}^{t} h_i^{1/p} e_i\Big\|^p \qquad \text{and}$$

$$(27) \qquad \sum_{i=1}^{t} h_i a_i + \sum_{k=0}^{n-1} \eta'_k(1-\eta'_k)^{-1} \sum_{i \in H_k} h_i a_i \leqq (\|x\|^p + \|y\|^p)/2 \leqq 1.$$

Now fix k, $0 \leqq k \leqq n-1$. For every $i \in H_k$

$$\alpha_i - \beta_i \leqq \eta_{k+1}\alpha_i \qquad \text{hence}$$

$$h_i \geqq (1-\eta'_k)\alpha_i^p/2 \geqq (1-\eta'_k)(\alpha_i-\beta_i)^p/2\eta_{k+1}^p \tag{28}$$

consequently

$$\Big\|\sum_{i\in H_k} h_i^{1/p}e_i\Big\|^p \geqq (1-\eta'_k)2^{-1}\eta_{k+1}^{-p}\Big\|\sum_{i\in H_k}(\alpha_i-\beta_i)e_i\Big\|^p. \tag{29}$$

Since $\eta_{k+1}=\eta_k\varepsilon^{-1/n}$ and $10\geqq\varepsilon^{-1/n}\geqq 1$ provided $\varepsilon\leqq 10^{-1}$ we get that $\|\Sigma_{i\in H_k}h_i^{1/p}e_i\|\geqq C_6\eta_k^{-p}\varepsilon_k^p$. Hence by Lemma 7,

$$\sum_{i\in H_k} h_i a_i \geqq C_7(M(n_k^p\varepsilon_k^{-p}))^{-1}. \tag{30}$$

Using (19), (26), (27) and (30) we deduce that

$$\|(x+y)/2\|^p \leqq 1 - C_8\sum_{k=0}^{n-1}\eta_k^2(M(\eta_k^p\varepsilon_k^{-p}))^{-1}.$$

By Lemma 6 (II), $(M(x^{-1}))^{-1}$ is equivalent to an increasing convex function $G(x)$ $0<x\leqq 1$. Using (21) and the facts that $p\geqq 2$ and $\Sigma_{k=0}^{n-1}\eta_k^2>\eta_{n-1}^2\geqq(\varepsilon^{1/n}/4)^2\geqq C_9$ we get that

$$\|(x+y)/2\|^p \leqq 1 - C_{10}G\Big(\sum_{k=0}^{n-1}\eta_k^{2-p}\varepsilon_k^p\Big) \leqq 1 - C_{11}G\Big(\sum_{k=0}^{n-1}\varepsilon_k^p\Big)$$

$$\leqq 1 - C_{12}G(\varepsilon^p) \leqq 1 - C_{13}(M(\varepsilon^{-p}))^{-1}.$$

This proves the first inequality in (9). To prove the other inequality in (9) let $\varepsilon>0$ and define $x=\Sigma\alpha_ie_i$, $y=\Sigma\beta_ie_i$, by

$$\alpha_i=\begin{cases}(S(2^mn))^{-1/p} & i\leqq 2^mn\\ 0 & i>2^mn\end{cases} \qquad \beta_i=\begin{cases}(S(2^mn))^{-1/p} & i\leqq(2^m-1)n\\ -(S(2^mn))^{-1/p} & (2^m-1)\ \ n<i\leqq 2^mn\\ 0 & i>2^mn\end{cases}$$

where $2\geqq g(2^m)\varepsilon^p\geqq 1$ and n is chosen such that $2g(2^mn)\geqq S(2^mn)/S(n)$. Clearly $\|x\|=\|y\|=1$, $\|x-y\|^p=2^pS(n)/S(2^mn)\geqq\varepsilon^p$ and $\|(x+y)/2\|^p=(S((2^m-1)n))/S(2^mn)=1-(S(2^mn)-S((2^m-1)n))/S(2^mn)\geqq 1-2^{-m}=1-C_{14}(M(\varepsilon^{-p}))^{-1}$. This concludes the proof of the theorem.

Let $d(a,p)$ be a Lorentz sequence space and suppose that there exists an n_0 such that $S(2n)/S(n)$ is an increasing function of n for $n \geqq n_0$. Then for $n > n_0$, $S(2^m n)/S(n) \geqq S(2^m n_0)/S(n_0) \geqq S(2^m)/S(n_0)$. Hence there exist constants A_1 and A_2 such that $A_1 S(2^m) \geqq g(2^m) \geqq A_2 S(2^m)$.

For example if $a_n \sim n^{-\alpha}(\log n)^{-\beta}$ $(0 \leqq \alpha < 1, 0 \leqq \beta)$, then $S(n) \sim n^{1-\alpha}(\log n)^{-\beta}$ and $S(2n)/S(n)$ is an increasing function of n. Hence $g(x) \sim x^{1-\alpha}(\log x)^{-\beta}$ and therefore $M(x) = g^{-1}(x) \sim x^{1/1-\alpha}(\log x)^{\beta/1-\alpha}$. Consequently, $\delta(\varepsilon) \sim (M(\varepsilon^{-p}))^{-1} \sim \varepsilon^{p/1-\alpha} |\log \varepsilon|^{-\beta/1-\alpha}$.

If the function $S(2n)/S(n)$ is decreasing then

$$k = \inf_n S(2n)/S(n) = \lim_n S(2n)/S(n) \quad \text{and}$$

$$S(2^m n)/S(n) \geqq k^m = (2^m)^{\log_2 k}.$$

Hence $g(x)$ is equivalent to the function $x^{\log_2 k}$, and $\delta(\varepsilon) \sim ((\varepsilon^{-p})^{1/\log_2 k})^{-1} = \varepsilon^{p/\log_2 k}$. For example, if $a_n \sim n^{-\alpha}(\log n)^{\beta}$ $(0 \leqq \alpha < 1, 0 < \beta)$, then $S(n) \sim n^{1-\alpha}(\log n)^{\beta}$ and $S(2n)/S(n)$ is a decreasing function of n. Since $\lim_n S(2n)/S(n) = 2^{1-\alpha}$ we get that $\delta(\varepsilon) \sim \varepsilon^{p/1-\alpha}$.

THEOREM 3. *Let X be a Banach space isomorphic to a Lorentz sequence space $d(a,p)$ $(p \geqq 2)$. Then there exists a constant $A > 0$ such that $\delta_X(\varepsilon)/\delta_d(\varepsilon) < A$ for all $0 < \varepsilon < 1$, where $\delta_X(\varepsilon)$ and $\delta_d(\varepsilon)$ are the moduli of convexity of X and $d(a,p)$ respectively.*

PROOF. Define $g(x)$ and $M(x)$ as in Theorem 2. Since $M(x)$ satisfies the Δ_2 condition for large x there is a constant C such that $M(2x) \leqq CM(x)$ for all $x \geqq 1$. Suppose that the theorem is false. Then there would exist a sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ such that for every i $\varepsilon_i \leqq 2^{-i}$ and

$$\delta_X(\varepsilon_i)/\delta_d(\varepsilon_i) \sim \delta_X(\varepsilon_i)M(\varepsilon_i^{-p}) \geqq C^i. \tag{31}$$

Choose now m_i so that $2^{-i-1} \leqq g(2^{m_i})\varepsilon_i^p \leqq 2^{-i}$ and n_i so that $2g(2^{m_i}) \geqq S(2^{m_i}n_i)/S(n_i)$.

For every integer i define the sequence $\{y_i^{(k)}\}_{k=1}^{2m_i}$ by

$$y_i^{(k)} = [\varepsilon_i/(S(n_i))^{1/p}] \sum_{t=l_i+(k-1)n_i+1}^{l_i+kn_i} e_t \qquad k = 1,2,\cdots,2^{m_i}$$

where

$$l_i = \sum_{j=1}^{i-1} 2^{m_j} n_j \qquad i = 2,3,\cdots$$

and $l_1 = 0$. Now $\| y_i^{(k)} \| = \varepsilon_i$, $k = 1,2,\cdots,2^{m_i}$ and

$$\Big\| \sum_{k=1}^{2^{m}} \pm y_i^{(k)} \Big\| = \varepsilon_i (S(2^{m_i} n_i))^{1/p} / S(n_i)^{1/p} \sim \varepsilon_i (g(2^{m_i}))^{1/p} \sim (2^{-i})^{1/p}.$$

Thus

$$\sum_{i=1}^{\infty} \Big\| \sum_{k=1}^{2^{m_i}} \pm y_i^{(k)} \Big\| \leqq \sum_{i=1}^{\infty} (2^{-i})^{1/p} < \infty$$

and therefore the series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{2^{m_i}} y_i^{(k)}$$

converges unconditionally. By a theorem of Kadec [5] this implies that

$$\sum_{i=1}^{\infty} \sum_{k=1}^{2^{m_i}} \delta_X(\| y_i^{(k)} \|) < \infty.$$

But by (31)

$$\sum_{k=1}^{2^{m_i}} \delta_X(\| y_i^{(k)} \|) \geqq 2^{m_i} \delta_X(\varepsilon_i) \geqq 2^{m_i} C^i / M(\varepsilon_i^{-p})$$

$$\geqq 2^{m_i} C^i / M(g(2^{m_i}) \cdot 2^i) \geqq 2^{m_i} C^i / C^i M(g(2^{m_i})) = 1,$$

a contradiction.

THEOREM 4. *Let $f(x)$ be a function defined on $[0,1]$, and let $p \geqq 2$. Necessary and sufficient conditions for $f(\varepsilon^p)$ to be equivalent to the modulus of convexity of some Lorentz sequence space $d(a,p)$ are:*

(i) *$f(0) = 0$ and $f(x)$ is equivalent to an increasing convex function,*

(ii) *There exists a constant $C > 0$ such that for all $0 < \lambda, \eta < 1$ $f(\lambda\eta) \geqq Cf(\lambda)f(\eta)$.*

PROOF. The conditions are necessary. By Theorem 2, $\delta(\varepsilon)$, the modulus of convexity of $d(a,p)$ ($\infty > p \geqq 2$), is equivalent to the function $f(\varepsilon^p)$ where

$$f(\varepsilon) = M(\varepsilon^{-1})^{-1}. \tag{32}$$

By the properties of the function $g(x) = M^{-1}(x)$ defined by (10), it is easily checked that $f(x)$ is equivalent to an increasing convex function satisfying the Δ_2 condition for small values of x. Hence (i) holds. Notice also that by (10) for all $n, m = 1,2,\cdots$

$$(33)\quad g(2^m 2^n) = \inf_k (S(2^m 2^n k)/S(k))$$

$$\geqq \inf_k (S(2^m 2^n k)/S(2^n k)) \inf_k (S(2^n k)/S(k)) \geqq g(2^m) g(2^n).$$

Since g is concave there exists therefore a $C_1 > 0$ such that for all $x, y, \geqq 1$

$$(34)\qquad g(xy) \geqq C_1 g(x) g(y)$$

and this implies that (ii) holds. Conversely, let $f(x)$ be an increasing convex function on $[0,1]$, satisfying $f(0) = 0$. Define $h(x) = (f(x^{-1}))^{-1}$ $(1 \leqq x < \infty)$. Since $f(x)/x$ is increasing the same is true for $h(x)/x$. By (ii) we have $h(xy) \leqq C_2 h(x) h(y)$ $(1 \leqq x, y < \infty)$ and thus in particular h satisfies the Δ_2 condition on $[1, \infty)$. Hence by Lemma 6 (I), $h(x)$ is equivalent to an increasing convex function $u(x)$ on $[1, \infty)$ with $u(1) = 1$. Define now $S(x) = u^{-1}(x)$ and

$$(35)\qquad a_n = \begin{cases} 1 & n = 1 \\ S(n) - S(n-1) & n > 1, \quad n \in N. \end{cases}$$

Since $S(x) = u^{-1}(x)$ is a concave function, it follows that the sequence a_n decreases to 0. Also by (ii), $S(xy) \geqq C_3 S(x) S(y)$ for $x, y \geqq 1$, and hence $g(x)$ defined by (10) is equivalent to $S(x)$. By our construction of $S(x)$ it follows therefore that the modulus of convexity of $d(a,p)$ where $\{a_n\}$ are given by (35) is equivalent to $f(\varepsilon^p)$ where f is the given function.

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